

# Number of binomial coefficients divided by a fixed power of a prime

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## Abstract

We state a general formula for the number of binomial coefficients  $n$  choose  $k$  that are divided by a fixed power of a prime  $p$ , i.e., the number of binomial coefficients divided by  $p^j$  and not divided by  $p^{j+1}$ .

Let  $n$  be a natural number and  $p$  be a prime. Let  $\theta_j(n)$  denote the number of binomial coefficients  ${}_nC_k = \binom{n}{k}$  such that  $p^j$  divides  ${}_nC_k$  and  $p^{j+1}$  does not divide  ${}_nC_k$ .

We represent  $n$  in the base  $p$ :  $n = c_0 + c_1p + c_2p^2 + \cdots + c_rp^r$ ,  $0 \leq c_i < p$ ,  $i = 0, 1, \dots, r$ ,  $c_r \neq 0$ .

Let  $W$  be the set of  $r$ -bit binary words, i.e.,

$$W = \{\mathbf{w} = w_1w_2 \dots w_r : w_i \in \{0, 1\}, 1 \leq i \leq r\}.$$

We partition  $W$  into  $r+1$  subsets  $W_j$ ,  $0 \leq j \leq r$ , where

$$W_j = \left\{ \mathbf{w} \in W : \sum_{i=1}^r w_i = j \right\}.$$

We define the functions  $F(\mathbf{w})$ ,  $L(\mathbf{w})$ , and  $M(\mathbf{w}, i)$  as follows:

$$F(\mathbf{w}) = \begin{cases} c_0 + 1 & \text{if } w_1 = 0, \\ p - c_0 - 1 & \text{if } w_1 = 1, \end{cases}$$

$$L(\mathbf{w}) = \begin{cases} c_r + 1 & \text{if } w_r = 0, \\ c_r & \text{if } w_r = 1, \end{cases}$$

$$M(\mathbf{w}, i) = \begin{cases} c_i + 1 & \text{if } w_i = 0 \text{ and } w_{i+1} = 0, \\ p - c_i - 1 & \text{if } w_i = 0 \text{ and } w_{i+1} = 1, \\ c_i & \text{if } w_i = 1 \text{ and } w_{i+1} = 0, \\ p - c_i & \text{if } w_i = 1 \text{ and } w_{i+1} = 1. \end{cases}$$

The general formula for  $\theta_j(n)$  is

$$\theta_j(n) = \sum_{\mathbf{w} \in W_j} F(\mathbf{w})L(\mathbf{w}) \prod_{i=1}^{r-1} M(\mathbf{w}, i). \quad (1)$$

Obviously, we have the sum of  ${}_r C_j$  terms and each term is the product of  $r+1$  factors. It is easy to establish that if  $p - c_\ell - 1 = 0$  for some  $\ell$ , then  $p - c_i - 1 = 0$  for all  $i \leq \ell$  (and some terms may vanish from the sum). It is also easy to establish that if  $p - c_\ell - 1 = 1$  for some  $\ell$ , then  $p - c_i - 1 = 1$  for all  $i \leq \ell$  (and the number of contributing factors in some terms is reduced). This means that the formula can be simplified for  $n$  of certain special forms.

Formula (1) reproduces known formulas for some particular values. For example, for  $j = 0$ , we obtain the known formula [1]

$$\begin{aligned} \theta_0(n) &= (c_0 + 1)(c_r + 1)(c_1 + 1) \cdots (c_{r-1} + 1) \\ &= (c_0 + 1)(c_1 + 1) \cdots (c_r + 1). \end{aligned}$$

For  $j = 1$ , we obtain the known formula [2]

$$\begin{aligned} \theta_1(n) &= (c_0 + 1)c_r(c_1 + 1)(c_2 + 1) \cdots (c_{r-2} + 1)(p - c_{r-1} - 1) \\ &\quad + (c_0 + 1)(c_r + 1)(c_1 + 1)(c_2 + 1) \cdots (p - c_{r-2} - 1)c_{r-1} \\ &\quad + \cdots \\ &\quad + (c_0 + 1)(c_r + 1)(p - c_1 - 1)c_2(c_3 + 1) \cdots (c_{r-1} + 1) \\ &\quad + (p - c_0 - 1)(c_r + 1)c_1(c_2 + 1) \cdots (c_{r-1} + 1) \\ &= \sum_{k=0}^{r-1} (c_0 + 1) \cdots (c_{k-1} + 1)(p - c_k - 1)c_{k+1}(c + k + 2 + 1) \cdots (c_r + 1). \end{aligned}$$

Other particular formulas for  $\theta_j(n)$  can be found in [3] and [4].

## References

- [1] Fine, N. J.: Binomial coefficients modulo a prime. *Amer. Math. Monthly* **54**, 589–592 (1947)
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